Yeditepe University
Math 160 Introductory Calculus
02nd January 2017, Final Exam
Solution Key

Name & Surname :……………………………………….. ….. Student No:………………………………..

Show all your work clearly. Answers without justifications and explicit calculations will get zero points. Mobile phones are strictly prohibidden. No extra papers are allowed, use back cover of pages to complete your solutions for extra calculations. There are 4 problems.

1- (15 pts) Two numbers have sum 15. What are the numbers if the product of the cube of one and the square of the other is as large as possible?

$Let\ x, y \in \mathbb{R}\ and\ x + y = 15,$

$P(x, y) = x^3\ y^2\ where\ y = 15 - x$

$So, \qquad P(x) = x^3\ (15 - x)^2$

$P'(x) = 3x^2\ (15 - x)^2 - 2x^3\ (15 - x) \quad = 0$

$3x^2\ (15 - x)\ \ [3(15 - x) - 2x] = 0$

$3x^2\ (15 - x)\ \ [45 - 5x] = 0$

$x = 0, x = 15\ and\ x = 9\ are\ critical\ points\ and$
$it\ is\ obvious\ that\ the\ largest\ value\ of\ P(x)\ is\ at\ x = 9\,.$
$Therefore; the\ numbers\ are\ x = 9\ and\ y = 6.$

2- (15+15=30 pts) Solve the following integrals

$$a)\quad \int \frac{x-2}{x^2+x}dx$$

$$b)\quad \int \frac{1}{\sqrt{9+x^2}}dx$$

$a)$ *Use integration of rational fractions method*

$$\frac{x-2}{x^2+x} = \frac{x-2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$A(x+1) + Bx = x - 2$$
$$A = -2$$
$$A + B = 1 \rightarrow B = 3$$

$$\int \frac{x-2}{x^2+x}dx = \int \left(\frac{-2}{x} + \frac{3}{x+1}\right)dx = -2ln|x| + 3ln|x+1| + c$$

$b)$ *Use inverse trigonometric substitution method*
Let $x = 3tan\theta$ *then* $dx = 3sec^2\theta d\theta$.

$$I = \int \frac{1}{\sqrt{9+x^2}}dx$$

$$= \int \frac{3sec^2\theta d\theta}{\sqrt{9+9tan^2\theta}} = \int \frac{3sec^2\theta d\theta}{\sqrt{9sec^2\theta}} = \int \frac{3sec^2\theta d\theta}{3sec\theta}$$

$$= \int sec\theta d\theta = ln|sec\theta + tan\theta| + c$$

$$We\ know\ that\ tan\theta = \frac{x}{3}\ and\ sec\theta = \frac{\sqrt{9+x^2}}{3}.$$

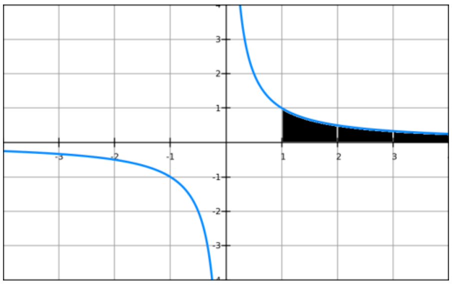
$$Hence;\ I = ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + c.$$

3- (15+15=30 pts) Find the areas of the regions bounded by the given curves

$$a)\ above\ y = 0, below\ y = \frac{1}{x}\ and\ to\ the\ right\ of\ x = 1.$$

$$b)\ y = x\ and\ \ y = x^2 - x.$$

a)

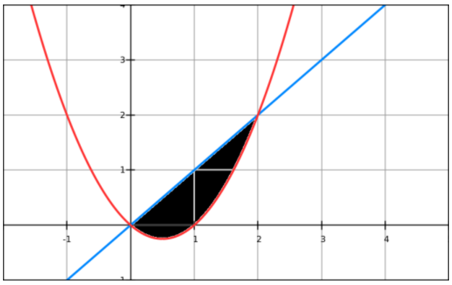


$$A = \int_1^{\infty} \frac{1}{x} dx$$

$$= \lim_{R\to\infty} \int_1^{R} \frac{1}{x} dx = \lim_{R\to\infty} lnx \Big|_1^R$$

$$= \lim_{R\to\infty} lnR = \infty$$

b)



$$x = x^2 - x$$

$$x^2 - 2x = x(x-2) = 0$$

$$x = 0\ and\ x = 2\ are\ intersection\ points.$$

$$Area\ is\ A = \int_0^2 (x - x^2 + x)\, dx$$

$$= \left[x^2 - \frac{x^3}{3}\right]\Big|_0^2 = \frac{4}{3}$$

4- (15+10=25 pts) Find the following limits

$$a)\ \lim_{x\to 0}\frac{1}{x}\int_{2}^{2+x}\sqrt{1+t^2}\,dt$$

$$b)\ \lim_{x\to\frac{\pi}{2}}\frac{ln(sinx)}{cosx}$$

$$a)\ \lim_{x\to 0}\frac{1}{x}\int_{2}^{2+x}\sqrt{1+t^2}\,dt = \lim_{x\to 0}\frac{\int_{2}^{2+x}\sqrt{1+t^2}\,dt}{x} \qquad \left[\frac{0}{0}\right]\ use\ L'hospital\ and\ F.T.C$$

$$= \lim_{x\to 0}\frac{\sqrt{1+(2+x)^2}}{1} = \sqrt{5}$$

$$b)\ \lim_{x\to\frac{\pi}{2}}\frac{ln(sinx)}{cosx} \qquad \left[\frac{0}{0}\right]\ use\ L'hospital$$

$$\lim_{x\to\frac{\pi}{2}}\frac{\frac{cosx}{sinx}}{-sinx} = \lim_{x\to\frac{\pi}{2}}\frac{cosx}{-sin^2x} = 0$$