Yeditepe University
Math 160 Introductory Calculus
24th October 2016, Midterm Exam Solution Key

Name & Surname :…………………………………………. ….. Student No:……………………………….

Show all your work clearly. Answers without justifications and explicit calculations will get zero points. Mobile phones are strictly prohibidden. No extra papers are allowed, use back cover of pages to complete your solutions for extra calculations. There are 5 problems.

1- (20 pts) Show that $\frac{b-a}{b} < ln\frac{b}{a} < \frac{b-a}{a}$ for $0 < a < b$.

Let $f(x) = lnx$ *be a differentiable function on* (a, b).

There exists a number $c \in (a,b)$ *such that* $f'(c) = \dfrac{f(b)-f(a)}{b-a} \rightarrow \dfrac{1}{c} = \dfrac{lnb - lna}{b-a}$ *by MVT.*

It is obvious that $\dfrac{1}{b} < \dfrac{1}{c} < \dfrac{1}{a}$ *. So;* $\dfrac{1}{b} < \dfrac{lnb-lna}{b-a} < \dfrac{1}{a} \rightarrow \dfrac{b-a}{b} < lnb - lna < \dfrac{b-a}{a}$

2- (10+10=20 pts) Let

$$f(x) = \begin{cases} cx^2+4 & if\ x \le 1 \\ 4cx-2 & if\ x > 1 \end{cases}$$

a) Find the value of c so that $f(x)$ is continuous for all x.
b) With your choice c found in part (a) show that $f(x)$ is not differentiable at $x = 1$.

a) From the definition of continuous function with one variable

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(cx^2+4) = c+4$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(4cx-2) = 4c-2$$

$$f(1) = c+4$$

$$c+4 = 4c-2 \ \rightarrow c = 2$$

$$b)\ f(x) = \begin{cases} 2x^2+4 & if\ x \le 1 \\ 8x-2 & if\ x > 1 \end{cases}$$

$$f'(1) = \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{f(1+h)-6}{h}$$
does not exist. Since;

$$\lim_{h\to0^+}\frac{f(1+h)-6}{h} \neq \lim_{h\to0^-}\frac{f(1+h)-6}{h} \quad where$$

$$\lim_{h\to0^+}\frac{f(1+h)-6}{h} = \lim_{h\to0^+}\frac{8h+8-2-6}{h} = 8 \ \ and$$
$$\lim_{h\to0^-}\frac{f(1+h)-6}{h} = \lim_{h\to0^-}\frac{2+4h+2h^2+4-6}{h} = 4.$$

Hence, $f(x)$ *is not differentiable at* $x = 1$

3- (10+10=20 pts) Find the derivatives of the following functions.

a) $y = sec^3(tanx)$

Let $tanx = u$ *then* $u' = sec^2(x)$.

$$y' = 3sec^2(u)\sec(u)\tan(u)\,u' = 3sec^3(u)\tan(u)u'$$

So; $y' = 3sec^3(tanx)\tan(tanx)sec^2(x)$.

b) $y = x^{cosx}$

$$lny = lnx^{cosx}$$

$$lny = \cos(x)lnx$$

$$\frac{y'}{y} = -\sin(x)\,lnx + \frac{1}{x}\cos(x)$$

$$y' = \left[-\sin(x)\,lnx + \frac{1}{x}\cos(x)\right]x^{cosx}$$

4- (10+10=20 pts) Find the limits of the following functions

a) $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$$\lim_{x\to 0} -x^2 = 0 \ \ and \ \ \lim_{x\to 0} x^2 = 0$$

$$So, \quad \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \ by \ Sandwich \ thm.$$

b) $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

$$= \lim_{x\to 0} \frac{\left(\sqrt{4+x}-2\right)\left(\sqrt{4+x}+2\right)}{x\left(\sqrt{4+x}+2\right)}$$

$$= \lim_{x\to 0} \frac{(4+x-4)}{x\left(\sqrt{4+x}+2\right)} = \frac{1}{4}$$

5- (5+5+5+5+10=30 pts) Let $f(x) = (x^3 - 1)^3$

a. Determine the intercepts of $f(x)$
b. Determine the critical points, inflection points and singular points.
c. Determine any asymptotes.
d. Determine the intervals where the function is increasing, decreasing, concave up and down
e. Sketch the graph of $f(x)$

a. x-int. (1,0) and y-int. (0,-1)

b. $y' = 3(x^3-1)^2 3x^2 = 9x^2(x^3-1)^2 = 0$. So, $x = 0\ (double\ root) and\ x = 1 (double\ root)\ are\ critical\ points.$
$y'' = 18(x^4 - x)(4x^3 - 1) = 0$. So; $x = 0, x = \frac{1}{\sqrt[3]{4}}\ and\ x = 1\ are\ inflection\ points.$
There is no singular points.

c. There is no asymptotes.

d.

x		-1	$\frac{1}{\sqrt[3]{4}}$	1
y'	+	+	+	+
y''	-	+	-	+
y	↗	↗	↗	↗
y	⌢	⌣	⌢	⌣

e.